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NON-CONSERVATIVELY LOADED STOCHASTIC COLUMNS

R. GANESAN and T. S. SANKAR
Concordia University, Montreal, Canada H3G 1M8

and

S. A. RAMU
Indian Institute of Science, Bangalore, India

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Abstract—A new finite element method is developed to analyse non-conservative structures with more than one parameter behaving in a stochastic manner. As a generalization, this paper treats the subsequent non-self-adjoint random eigenvalue problem that arises when the material property values of the non-conservative structural system have stochastic fluctuations resulting from manufacturing and measurement errors. The free vibration problems of stochastic Beck's column and stochastic Leipholz column whose Young's modulus and mass density are distributed stochastically are considered. The stochastic finite element method that is developed, is implemented to arrive at a random non-self-adjoint algebraic eigenvalue problem. The stochastic characteristics of eigensolutions are derived in terms of the stochastic material property variations. Numerical examples are given. It is demonstrated that, through this formulation, the finite element discretization need not be dependent on the characteristics of stochastic processes of the fluctuations in material property value.

1. INTRODUCTION

With the advancement of space mechanics and rocket propulsion systems, the study of the stability behavior of launch systems has gained much importance and dynamic analysis is being used to investigate their stability (Bolotin, 1964). In such structures, several uncertainties in structural parameters like length, moment of inertia, cross-sectional area, material parameters like elastic modulus, mass density, etc., and the loading parameters, are always present and unavoidable. A detailed investigation is necessary to assign the probability limits of various characteristics, such as amplitude excursions, peak and envelope statistics, etc., of the eigenvalues of the structural system. The usage of modern engineering materials, which are characterized by their inherent uncertainties, does necessitate such an analysis to ascertain reliable design and performance.

Literature surveys indicate that structural systems subjected to non-conservative loading in which the loading and system parameters are independent stochastic fields are not investigated so far. Only the type of systems such as columns subjected to random loadings and having random material properties, resulting in a conservative system were analysed by a number of authors (Collins and Thomson, 1969 ; Shinozuka and Astill, 1972 ; Hoshiya and Shah, 1971 ; Augusti *et al.*, 1981), using Euler's criterion, i.e. statical method. At the same time, Ariaratnam (1967) considered the stability of a deterministic column subjected to a random loading in time.

The stochastic finite element method in the field of structural analysis where finite element approaches are used in a probabilistic setting has received extensive attention recently with different methodologies (Contreras, 1980 ; Liu *et al.*, 1986, 1988 ; Yamazaki *et al.*, 1985 ; Shinozuka and Deodatis, 1988 ; Vanmarcke and Grigoriu, 1983 ; Spanos and Ghanem, 1989). An excellent review of the research efforts in stochastic finite elements is given by Benaroya and Rehak (1988). As can be seen from this review article, the stochastic finite element methods fall into two major categories : (i) methods for response moment calculations, and (ii) methods for reliability calculations. This is so because, if the random fields are employed, the formulation leading to reliability evaluation needs to be entirely different from that of the response moment calculations. If random variables are employed as in the works of Liaw and Yang (1991), the methods leading to the evaluation of response moments can also be modified to evaluate the reliability. Nakagiri *et al.* (1987) used a version of the stochastic finite element method to investigate the eigenvalue problem of laminated composite plates considering stochastic variation of stiffness. In all the works published so far, the system parameters are described either directly as random variables or as stochastic fields converted to equivalent random variables defined at the centroid of the finite element. The shortcomings of such a description are immediately obvious; to cite just a few:

(a) the use of only the mean and variance in the description of a random variable is inadequate to describe a stochastic field for which a unique autocorrelation function or scale of fluctuation is also necessary;

(b) the size of the finite element becomes a function of the stochastic property of the field rather than being governed by the usual considerations of the deterministic FEM;

(c) finite element discretization when multiple stochastic fields are to be represented cannot be considered as appropriate;

(d) the perturbation method employed in all the above publications leads to complicated recursive equations and can be avoided.

Even the recent work by Liaw and Yang (1991), in which the non-conservative loading case is also addressed, has the above stated shortcomings. The present authors have proposed and used a new version of the stochastic finite element method which is free from all the above shortcomings for both self-adjoint and non-self-adjoint problems (Ramu and Ganesan, 1991, 1992a,b; Sankar *et al.*, 1992a) and also for singularity problems (Sankar *et al.*, 1992b). Here, the past investigations by the present authors have been extended and used for solving the free vibration problem of non-self-adjoint type which arises when the material property such as Young's modulus, mass density of the non-conservative systems are random processes in space. In other words, Beck's column and Leipholz's column whose Young's modulus and mass per unit length have a stochastic variation are investigated. The stochastic variation of Young's modulus and mass per unit length are considered to be spatially distributed one-dimensional, univariate stochastic fields.

2. DESCRIPTION OF THE PROBLEM

Consider a free-free column as shown in Fig. 1 which is subjected to two end loads P at $x = 0$ inclined at an angle of $\alpha_0 \phi_0$ to the undeformed axis of the column, where ϕ_0 is the angle between the tangent to the deformed axis of the column and the undeformed axis of the column at $x = 0$ and $(P+Q)$ at $x = l$, inclined at an angle of $\alpha_1 \phi_1$, where ϕ_1 is the angle between the tangent to the deformed axis and the undeformed axis of the column at $x = l$.

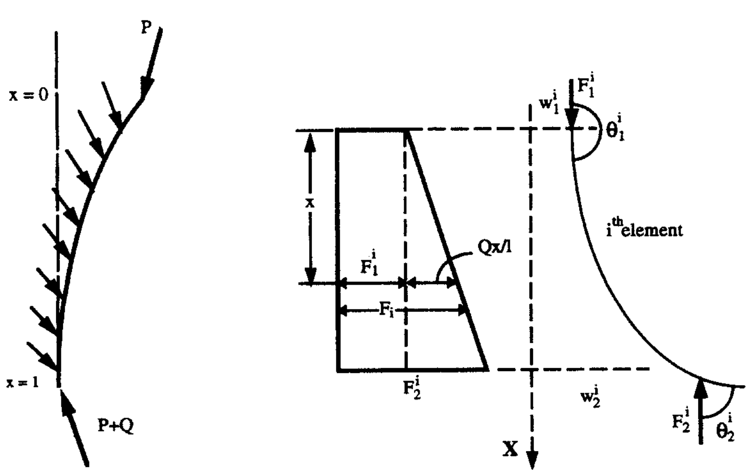

Fig. 1. Column under general non-conservative loading.

Also, a distributed follower load of uniform intensity p/unit length inclined at the angle $\alpha_D\phi$ with respect to the undeformed axis of the column is considered, where ϕ is the angle made by the tangent to the deformed axis at any arbitrary point with the undeformed axis of the column. Here x is measured from the end from which the distributed load is directed towards and α_0, α_1 and α_D define the degrees of the non-conservativeness of the respective forces. The loading configuration represented is analogous to force distributions arising in non-conservative problems discussed by earlier investigators.

3. STOCHASTIC CHARACTERIZATION OF THE SYSTEM PARAMETERS

The Young's modulus and mass density are distributed randomly, along the undeformed axis of the column. The fluctuations over their mean values are assumed to constitute independent one-dimensional, univariate, homogeneous, real, spatial stochastic fields. The Young's modulus and mass density can thus be expressed by

$$E(x) = \bar{E}[1+a(x)], \tag{1}$$

$$m(x) = \bar{m}[1+b(x)], \tag{2}$$

where $\bar{E}$ and $\bar{m}$ are the mean values of the Young's modulus and mass density respectively. $a(x)$ and $b(x)$ are independent, one-dimensional, univariate, homogeneous, real, spatial stochastic fields. The processes are characterized by their respective autocorrelation functions R_{aa} and R_{bb} (or by their equivalent power spectral density functions S_{aa} and S_{bb}) and scale of fluctuations Θ_E and Θ_m. The variances are σ_E^2 and σ_m^2, respectively. The autocorrelation functions are, by definition, given by

$$R_{aa}(\xi) = \langle a(x)\cdot a(x+\xi)\rangle \tag{3}$$

$$R_{bb}(\xi) = \langle b(x)\cdot b(x+\xi)\rangle. \tag{4}$$

4. GENERAL CONSISTENT FINITE ELEMENT FORMULATION

Since ϕ_0, ϕ_1 and ϕ are small, the cosines of these angles may be taken to approach unity. Therefore, for equilibrium with respect to forces parallel to the undeformed axis of the column one gets $Q = pl$.

The column is now discretized into n elements. Let the ith element be considered. The nodal degrees of freedom are taken to be w_1^i, Θ_1^i, w_2^i, and Θ_2^i for the ith element. w_1^i, w_2^i are the transverse deflections of the two ends of the element and Θ_1^i and Θ_2^i are the rotations of the tangents to the deformed axis of the element with respect to the undeformed axis. The transverse displacement $w(x, t)$ at any arbitrary point of this element is given by

$$w(x, t) = N^i q^i, \tag{5}$$

where

$$N^i = \{N_1\, N_2\, N_3\, N_4\}^i,$$
$$q_i^T = \{w_1, \Theta_1, w_2, \Theta_2\}^i.$$

N^i is a row vector of interpolation polynomials (cubic Hermitian polynomials) which are the same as that of a corresponding deterministic structure. The components of q_i^T are functions of t, the time variable.

If T^i represents the kinetic energy stored in the ith element, then

$$T^i = \frac{1}{2}\int_0^{l_i} m(x)A_i\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x$$

$$= \tfrac{1}{2}q_t^{i^{\mathrm{T}}}\left[\int_0^{l_i} N^{i^{\mathrm{T}}}\bar{m}(1+b(x))A_iN^i\,\mathrm{d}x\right]q_t^i$$

$$= \tfrac{1}{2}q_t^{i^{\mathrm{T}}}M^iq_t^i, \tag{6}$$

where $M^i = \bar{M}^i + M^i(\Omega)$, and the differentiation w.r.t. time is indicated by the subscript t.

The elements of $\bar{M}^i$ are given by

$$\bar{m}_{ij} = \int_0^{l_i} \bar{m}AN_iN_j\,\mathrm{d}x \tag{7}$$

and

$$m_{ij}(\Omega) = \bar{m}A\int_0^{l_i} b(x)N_iN_j\,\mathrm{d}x,$$

$$m_{ij} = \bar{m}_{ij} + m_{ij}(\Omega). \tag{8}$$

If U^i represents the strain energy stored in the ith element, then

$$U^i = \frac{1}{2}\int_0^{l_i} \bar{E}(1+a(x))I_i\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x$$

$$= \tfrac{1}{2}q_i^{\mathrm{T}}\left[\int_0^{l_i} N''^{i^{\mathrm{T}}}\bar{E}(1+a(x))I_iN''^i\,\mathrm{d}x\right]q_i$$

$$= \tfrac{1}{2}q^{i^{\mathrm{T}}}K^iq^i. \tag{9}$$

Here primes indicate differentiation w.r.t. x and

$$K^i = \int_0^{l_i} N''^{i^{\mathrm{T}}}\bar{E}(1+a(x))I_iN''^i\,\mathrm{d}x$$

$$= \bar{K}^i + K^i(\Omega).$$

The coefficients of the stiffness matrix are thus given by

$$k_{ij} = \int_0^{l_i} N_i''\bar{E}(1+a(x))I_iN_j''\,\mathrm{d}x$$

$$= \bar{E}I_i\int_0^{l_i} N_i''N_j''\,\mathrm{d}x + \bar{E}I_i\int_0^{l_i}(a(x))N_i''N_j''\,\mathrm{d}x$$

$$= \bar{k}_{ij} + k_{ij}(\Omega). \tag{10}$$

Because of the presence of the axial component of the distributed follower load, the ith element is subjected to a uniformly varying axial compression increasing from F_1^i to F_2^i (refer to Fig. 1). The axial compression at any arbitrary section of the element is given by $(F_1^i + px) = F_1^i + (Q/l)x$, where x is measured to be positive in the increasing direction of the axial compression, i.e. from end 1 to end 2. Therefore, the work done by the axial compression in the element W_c^i is given by

$$W_c^i = \frac{1}{2}\int_0^{l_i} (F_1^i + Qx/l)\left(\frac{\partial w}{\partial x}\right)^2 dx$$

$$= \tfrac{1}{2}F_1^i q^{i^T}\left[\int_0^{l_i} N'^{i^T} N'^i \, dx\right] q^i + \tfrac{1}{2}Q q^{i^T}\left[\int_0^{l_i} N'^{i^T}(L_2 l_i/l) N'^i \, dx\right] q^i$$

$$= \tfrac{1}{2}F_1^i q^{i^T} k_{GC}^i q^i + \tfrac{1}{2}Q q^{i^T} k_{GD}^i q^i, \tag{11}$$

where

$$k_{GC}^i = \int_0^{l_i} N'^{i^T} N'^i \, dx \tag{12}$$

and

$$k_{GD}^i = \int_0^{l_i} N'^{i^T}(L_2 l_i/l) N'^i \, dx, \tag{13}$$

where L_1 and L_2 are the natural coordinates.

k_{GC}^i (which is actually the geometric stiffness matrix used in the stability analysis when the compressive force is constant) and k_{GD}^i (which is the geometric stiffness matrix which accounts for the linearly varying compression force starting from a value zero at one end) are deterministic coefficients.

It may be noted that the first term in the above expression gives the work done by the constant part of the compression F_1^i for the element, whereas the second term is due to the work done by the uniformly varying compression increasing from 0 to $(F_2^i - F_1^i)$ as shown in Fig 1.

The lateral components of the follower forces are dependent on the slopes of the tangents to the deformed axis of the column at the points of application of these forces and hence there is no unique scalar work function corresponding to these forces. Therefore, the virtual work done by these forces is appended as an external term in the usual form of Hamilton's principle.

The virtual work done by the non-conservative components of the forces is worked out considering the virtual displacements to be the variations of the actual displacements.

The virtual work δW_{DNC}^i done by the distributed follower force in the element i, is given by

$$\delta W_{DNC}^i = -\int_0^{l_i} p \sin(\alpha_D \phi)\delta w \, dx = \int_0^{l_i} p\alpha_D \frac{\partial w}{\partial x}\cdot \delta w \, dx$$

$$= p\alpha_D \delta q^{i^T}\left[\int_0^{l_i} N^{i^T} N'^i \, dx\right] q^i$$

$$= [\alpha_D Q/l]\delta q^{i^T} k_{DNC}^i q^i, \tag{14}$$

where

$$k_{DNC}^i = \int_0^{l_i} N^{i^T} N'^i \, dx$$

and subscript DNC represents the distributed non-conservative loading of the column.

The virtual work done by the concentrated follower force at the ends may be taken together as δW_{CNC} given by

$$\delta W_{\mathrm{CNC}} = -P\sin(\alpha_0\phi_0)\delta w_0 - (P+Q)\sin(\alpha_1\phi_1)\delta w_1$$
$$= P\alpha_0\left(\frac{\partial w}{\partial x}\right)_0 \delta w_0 - (P+Q)\alpha_1\left(\frac{\partial w}{\partial x}\right)_1 \delta w_1$$
$$= P\alpha_0\delta w_0\left(\frac{\partial w}{\partial x}\right)_0 - (P+Q)\alpha_1\delta w_1\left(\frac{\partial w}{\partial x}\right)_1. \quad (15)$$

Here subscript CNC represents the non-conservative loadings at the ends $x = 0$ and $x = l$ of the column.

Now, the total kinetic energy of the entire structure is

$$T = \sum_{i=1}^{n} T^i = \left(\sum_{i=1}^{n} \tfrac{1}{2}\dot{q}_t^{i^{\mathrm{T}}} M^i \dot{q}_t^i\right)$$
$$= \sum_{i=1}^{n} \tfrac{1}{2}\dot{q}_t^{i^{\mathrm{T}}}(\bar{M}^i + M^i(\Omega))\dot{q}_t^i. \quad (16)$$

The total strain energy is given by

$$U = \sum_{i=1}^{n} U^i = \left(\sum_{i=1}^{n} \tfrac{1}{2}q^{i^{\mathrm{T}}} K^i q^i\right)$$
$$= \sum_{i=1}^{n} \tfrac{1}{2}q^{i^{\mathrm{T}}}(\bar{K}^i + K^i(\Omega))q^i. \quad (17)$$

Total work done by the conservative part of the forces, W_c is given by

$$W_c = \sum_{i=1}^{n} W_c^i$$
$$= \sum_{i=1}^{n} \tfrac{1}{2}F_1^i q^{i^{\mathrm{T}}} K_{\mathrm{GC}}^i q^i + \sum_{i=1}^{n} \tfrac{1}{2}Q q^{i^{\mathrm{T}}} K_{\mathrm{GD}}^i q^i. \quad (18)$$

The total virtual work done by the non-conservative components of the forces can now be defined in the form,

$$\delta W_{\mathrm{NC}} = \sum_{i=1}^{n} \delta W_{\mathrm{DNC}}^i + \delta W_{\mathrm{CNC}}$$
$$= \sum_{i=1}^{n}\left\{(\alpha_{\mathrm{D}}Q/l)\delta q^{i^{\mathrm{T}}} K_{\mathrm{DNC}}^i q^i\right\} + P\alpha_0\delta W_0\left(\frac{\partial w}{\partial x}\right)_0 - (P+Q)\alpha_1\delta W_1\left(\frac{\partial w}{\partial x}\right)_1. \quad (19)$$

The classical Hamilton's principle,

$$\delta\int_{t_1}^{t_2} L\,\mathrm{d}t = 0 \quad (20)$$

if modified for non-conservative systems becomes

$$\int_{t_1}^{t_2} [\delta(T - U + W_c) + \delta W_{\mathrm{NC}}]\,\mathrm{d}t = 0. \quad (21)$$

Substituting for T, U, W_c and δW_{NC}, this becomes

$$\int_{t_1}^{t_2} \delta\left[\sum_{i=1}^{n} \tfrac{1}{2} q_t^{iT}(\bar{M}^i + M^i(\Omega)) q_t^i\right] dt - \int_{t_1}^{t_2} \delta\left[\sum_{i=1}^{n} \tfrac{1}{2} q^{iT}(\bar{K}^i + K^i(\Omega)) q^i + \sum_{i=1}^{n} F_1^i \tfrac{1}{2} q^{iT} K_{GC}^i q^i + \sum_{i=1}^{n} Q \tfrac{1}{2} q^{iT} K_{GD}^i q^i\right] dt + \int_{t_1}^{t_2} \left[\sum_{i=1}^{n} (\alpha_D Q/l) \delta q^{iT} K_{DNC}^i q^i + P\alpha_0 \delta W_0 \left(\frac{\partial w}{\partial x}\right)_0 + (P+Q)\alpha_1 \delta W_1 \left(\frac{\partial w}{\partial x}\right)_1\right] dt = 0. \quad (22)$$

The summations are carried out in the sense of finite element assemblage, taking the global displacement vector to be q. In addition, w_0 is taken as q_1, $(\partial w/\partial x)_0$ as q_2, w_1 as q_{2n+1} and $(\partial w/\partial x)_1$ as q_{2n+2}. Now, if contemporaneous variations of q are taken, while integrating the first term by parts, the δW_{NC} is given as follows:

$$\delta W_{NC} = \int_{t_1}^{t_2} [-\delta q^T M q_{tt} - \delta q^T K q + \delta q^T P K_{GC} q + (\delta q^T Q K_{GC}^* q + \delta q^T Q K_{GD} q) + \delta q^T (\alpha_D Q/l) K_{DNC} q + \delta q^T \alpha_0 P K_{CNC1} q - q^T \alpha_1 P K_{CNC2} q - q^T \alpha_1 Q K_{CNC2} q]\, dt = 0. \quad (23)$$

F_1^i in eqn (18) is put in the form:

$$F_1^i = P + \beta^i Q, \quad (24)$$

where $\beta^i < 1.0$ corresponds to the fraction of the total distributed load acting at the trailing node of the element i and

$$K_{GC}^{i*} = \beta^i K_{GC}^i. \quad (25)$$

Matrices K_{CNC1} and K_{CNC2} contain non-zero elements only in one location each, namely $(1,2)$ and $(2n+1, 2n+2)$; and therefore, these matrices need not be assembled but the corresponding terms in K_{GC} and K_{GC}^* may be modified by the appropriate addition of the non-zero terms in K_{CNC1} and K_{CNC2}, after multiplying them by the respective factors shown in eqn (23).

Substituting $q = \bar{q}^* e^{st}$ in eqn (23) and considering the arbitrariness of the variation of q gives

$$\{-s^2(\bar{M} + M(\Omega)) + (-\bar{K} - K(\Omega)) + P(K_{GC} + \alpha_0 K_{CNC1} - \alpha_1 K_{CNC2} + Q(K_{GC}^* + K_{GD}) + \alpha_D K_{DNC} - \alpha_1 K_{CNC2})\} \bar{q}^* e^{st} = 0, \quad (26)$$

i.e.

$$\{-s^2(\bar{M} + M(\Omega)) + (-\bar{K} - K(\Omega)) + P K_{GP}^* + Q K_{GQ}^*\} \bar{q}^* = 0, \quad (27)$$

where

$$K_{GP}^* = K_{GC} + \alpha_0 K_{CNC1} - \alpha_1 K_{CNC2},$$

$$K_{GQ}^* = K_{GC}^* + K_{GD} + \alpha_D K_{DNC} - \alpha_1 K_{CNC2}.$$

5. PARTICULAR CASES

5.1. *Beck's column*

This is the case of a cantilever column with a concentrated follower force P at the free end, giving

$Q = 0$ and $\alpha_1 = 0$, leading to

$$\{-s^2(\bar{M}+M(\Omega))+(-\bar{K}-K(\Omega))+PK^{*}_{GP}\}\bar{q}^{*}\,e^{st} = 0. \tag{28}$$

The geometric boundary conditions are imposed by making the corresponding rows and columns in the respective global matrices zero or by simply removing those rows and columns from the global matrices to get reduced matrices for the restrained structure. The latter method will reduce the number of operations for calculating the covariance matrix.

The eigenvalue problem is given by the equation

$$-(\bar{K}+K(\Omega)-P(K_{GC}+\alpha_0 K_{CNC1}))\bar{q}^{*} = s^2(\bar{M}+M(\Omega)). \tag{29}$$

This eigenvalue problem is solved for the eigenvalues s^2 for particular values of the parameter α_0 as the value of P is varied.

The onset of instability by divergence or by flutter is indicated by the appearance of s^2 values which are real, positive and complex, respectively, and the corresponding load P is the critical load at divergence or flutter, as the case may be. $\alpha_0 = 1$ is the classical Beck's column. The calculation of the eigenvalue statistics will give the required statistical information about Beck's flutter load.

5.2. *Leipholz column*

This is a cantilever column with the follower force uniformly distributed along its length. Here,

$$P = 0.$$

Contributions from K_{CNC1} and K_{CNC2} are taken to be equal to zero. The resulting equation is

$$\{-s^2(\bar{M}+M(\Omega))-(\bar{K}+K(\Omega))+Q(K^{*}_{GC}+\alpha_D K_{DNC})\}\bar{q}^{*}\,e^{st} = 0. \tag{30}$$

For a fully tangential distributed load the non-conservative parameter,

$$\alpha_D = 1.0.$$

Now, the geometric boundary conditions are imposed and the reduced matrices are obtained for the constrained structure which also results in the reduction of covariance matrix size and a reduced amount of summation.

6. STOCHASTIC CHARACTERIZATION OF THE INFLUENCE COEFFICIENTS

Let the stiffness coefficient k_{ij} now be considered. The mean and variance values of k_{ij} are derived as follows. The mean value is given by the following averaging procedure:

$$k_{ij} = \langle k_{ij}\rangle = \langle \bar{k}_{ij}+k_{ij}(\Omega)\rangle \tag{31}$$

$$= \left\langle \bar{E}I\int_0^{l_c} N_i''(x)\cdot N_j''(x)\,dx\right\rangle + \left\langle \bar{E}I\int_0^{l_c} a(x)N_i''(x)\cdot N_j''(x)\,dx\right\rangle \tag{32}$$

$$= \bar{E}I\int_0^{l_c} N_i''(x)\cdot N_j''(x)\,dx + \bar{E}I\int_0^{l_c}\langle a(x)\rangle N_i''(x)\cdot N_j''(x)\,dx$$

$$= \bar{E}I\int_0^{l_c} N_i''(x)\cdot N_j''(x)\,dx = \bar{k}_{ij} \tag{33}$$

as $\langle a(x)\rangle = 0$.

Similarly,

$$\text{Variance }(k_{ij}) = \text{var }(k_{ij})$$
$$= \text{Var }(\bar{k}_{ij}+k_{ij}(\Omega)) = \text{var }(k_{ij}(\Omega)) \tag{34}$$
$$= \left\langle \bar{E}I\int_0^{l_e} a(x)N_i''(x)\cdot N_j''(x)\,\mathrm{d}x\cdot \bar{E}I\int_0^{l_e} a(x)N_i''(x)\cdot N_j''(x)\,\mathrm{d}x\right\rangle$$
$$= \bar{E}^2I^2\int_0^{l_e}\int_0^{l_e}\langle a(\xi_1)\cdot a(\xi_2)\rangle\cdot N_i''(\xi_1)\cdot N_j''(\xi_1)\cdot N_i''(\xi_2)\cdot N_j''(\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \tag{35}$$
$$= \bar{E}^2I^2\int_0^{l_e}\int_0^{l_e} R_{aa}(\xi_1-\xi_2)\cdot N_i''(\xi_1)\cdot N_j''(\xi_1)\cdot N_i''(\xi_2)\cdot N_j''(\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2. \tag{36}$$

The mass coefficient m_{ij} is given by

$$m_{ij} = \int_0^{l_e} Am(x)\cdot N_i(x)\cdot N_j(x)\,\mathrm{d}x \tag{37}$$
$$= \bar{m}A\int_0^{l_e} N_i(x)\cdot N_j(x)\,\mathrm{d}x+\bar{m}A\int_0^{l_e} b(x)\cdot N_i(x)\cdot N_j(x)\,\mathrm{d}x \tag{38}$$
$$= \bar{m}_{ij}+m_{ij}(\Omega). \tag{39}$$

Similar analysis for mean and variance of mass coefficients will show that

$$\text{Mean of } m_{ij} = \bar{m}_{ij}$$
$$\text{Var }(m_{ij}) = \text{Var }(m_{ij}(\Omega))$$
$$= \bar{m}^2A^2\int_0^{l_e}\int_0^{l_e} R_{bb}(\xi_1-\xi_2)N_i(\xi_1)\cdot N_j(\xi_1)\cdot N_i(\xi_2)\cdot N_j(\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2. \tag{40}$$

As can be seen, the evaluation of the variances using the above expressions will be tedious. Also, the full correlation function expression is needed for the evaluation. Such adequate information is seldom available and the experimental data seldom allow one to distinguish among competing analytical models for the correlation function. So, a simplified treatment is now proposed. The stochastic processes $a(x)$, $b(x)$ are assumed to be characterized by three parameters in each case: the means, which are zero, the standard deviations σ_E, σ_m and the scale of fluctuations Θ_E, Θ_m. The second order properties are covered by either the autocorrelation function or power spectral densities or by the variance functions (Vanmarcke and Grigoriu, 1983). So, the fluctuating parts of stiffness and mass elements are evaluated using the local averages over elements of material property fluctuations. For the ith finite element, the local averages are

$$E_i = \frac{1}{l_i}\int_0^{l_i} a(x)\,\mathrm{d}x; \quad m_i = \frac{1}{l_i}\int_0^{l_i} b(x)\,\mathrm{d}x, \tag{41}$$

where l_i = length of the ith element.

The properties of local averages are then:

$$\langle E_i \rangle = \left\langle \frac{1}{l_i} \int_0^{l_i} a(x)\,\mathrm{d}x \right\rangle = \frac{1}{l_i} \int_0^{l_i} \langle a(x) \rangle\,\mathrm{d}x = 0,$$

$$\langle m_i \rangle = \left\langle \frac{1}{l_i} \int_0^{l_i} b(x)\,\mathrm{d}x \right\rangle = \frac{1}{l_i} \int_0^{l_i} \langle b(x) \rangle\,\mathrm{d}x = 0. \tag{42}$$

The variances of local averages are

$$\mathrm{Var}\,(E_i) = \left\langle \frac{1}{l_i} \int_0^{l_i} a(x)\,\mathrm{d}x \cdot \frac{1}{l_i} \int_0^{l_i} a(x)\,\mathrm{d}x \right\rangle = \frac{1}{l_i^2} \int_0^{l_i} \int_0^{l_i} \langle a(\xi_1) \cdot a(\xi_2) \rangle\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2, \tag{43}$$

$$\mathrm{Var}\,(m_i) = \left\langle \frac{1}{l_i} \int_0^{l_i} b(x)\,\mathrm{d}x \cdot \frac{1}{l_i} \int_0^{l_i} b(x)\,\mathrm{d}x \right\rangle = \frac{1}{l_i^2} \int_0^{l_i} \int_0^{l_i} \langle b(\xi_1) \cdot b(\xi_2) \rangle\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2. \tag{44}$$

Since $\langle a(\xi_1) \cdot a(\xi_2) \rangle = R_{aa}(\xi_1 - \xi_2)$ and $\langle b(\xi_1) \cdot b(\xi_2) \rangle = R_{bb}(\xi_1 - \xi_2)$, the above equations can be written as

$$\mathrm{Var}\,(E_i) = \frac{1}{l_i^2} \int_0^{l_i} \int_0^{l_i} R_{aa}(\xi_1 - \xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \tag{45}$$

and

$$\mathrm{Var}\,(m_i) = \frac{1}{l_i^2} \int_0^{l_i} \int_0^{l_i} R_{bb}(\xi_1 - \xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2. \tag{46}$$

In terms of variance functions, which characterize the dependence of variance of local average on the size of the element, the above variance expressions can be written as,

$$\mathrm{Var}\,(E_i) = \sigma_{\mathrm{E}}^2 \gamma_{\mathrm{E}}(l_i) = \sigma_{\mathrm{E}}^2 \cdot \frac{\Theta_{\mathrm{E}}}{l_i}; \quad l_i \gg \Theta_{\mathrm{E}}, \Theta_{\mathrm{M}} \tag{47}$$

and

$$\mathrm{Var}\,(m_i) = \sigma_{\mathrm{m}}^2 \gamma_{\mathrm{m}}(l_i) = \sigma_{\mathrm{m}}^2 \cdot \frac{\Theta_{\mathrm{m}}}{l_i}; \quad l_i \gg \Theta_{\mathrm{E}}, \Theta_{\mathrm{M}}, \tag{48}$$

where $\gamma_{\mathrm{E}}(l_i)$, $\gamma_{\mathrm{m}}(l_i)$ are the variance functions of spatial averages of E and m respectively and Θ_{E}, Θ_{m} are the scales of fluctuation of E and m, respectively. The covariance functions of stiffness coefficients can now be calculated for any two coefficients $k_{ij}(\Omega)$ and $k_{rs}(\Omega)$, where i, j and r, s are the nodal point indices for those two elements through the use of spatial averages and are given by

$$\mathrm{Var}\,(E_i) = \begin{cases} \sigma_{\mathrm{E}}^2 & l_i \ll \Theta_{\mathrm{E}}, \\ \sigma_{\mathrm{E}}^2 \dfrac{\Theta_{\mathrm{E}}}{l_i} & l_i \gg \Theta_{\mathrm{E}} \end{cases} \tag{49}$$

and

$$\mathrm{Var}\,(m_i) = \begin{cases} \sigma_{\mathrm{m}}^2 & l_i \ll \Theta_{\mathrm{E}}, \\ \sigma_{\mathrm{m}}^2 \dfrac{\Theta_{\mathrm{m}}}{l_i} & l_i \gg \Theta_{\mathrm{E}}. \end{cases} \tag{50}$$

The cross covariances between stiffness and mass fluctuation components are zero as the two stochastic processes $a(x)$, $b(x)$ which describe the fluctuations around the mean values need not be dependent. Alternatively, the independence between the two is implied here. The covariance functions of stiffness coefficients are calculated for any two coefficients $k_{ij}(\Omega)$ and $k_{rs}(\Omega)$ of the overall system matrices:

$$\langle k_{ij}(\Omega)\cdot k_{rs}(\Omega)\rangle = \left\langle \bar{E}I\int_0^{l_1} a(x)N_i''(x)\cdot N_j''(x)\,\mathrm{d}x\cdot \bar{E}I\int_0^{l_2} a(x)N_r''(x)\cdot N_s''(x)\,\mathrm{d}x\right\rangle, \quad (51)$$

where l_1 and l_2 are the lengths of two elements, i.e. distances between nodes i and j and nodes r and s,

$$\langle k_{ij}(\Omega)\cdot k_{rs}(\Omega)\rangle = \bar{E}^2I^2\left\langle \int_0^{l_1}\int_0^{l_2} a(\xi_1)a(\xi_2)N_i''(\xi_i)\cdot N_j''(\xi_2)N_r''(\xi_1)\cdot N_s''(\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\right\rangle. \quad (52)$$

The above equation is approximated using the spatial averages as,

$$\left\{\bar{E}^2I^2\int_0^{l_i}\int_0^{l_i} N_i''(\xi_1)\cdot N_j''(\xi_2)N_r''(\xi_1)\cdot N_s''(\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\right\}\cdot\langle E_1(ij)\cdot E_2(rs)\rangle, \quad (53)$$

which for equal sized finite elements becomes

$$\left\{\bar{E}^2I^2\int_0^{l} N_i''(x)N_j''(x)\,\mathrm{d}x\cdot\int_0^{l} N_r''(x)N_s''(x)\,\mathrm{d}x\right\}\cdot\langle E_1(ij)\cdot E_2(rs)\rangle. \quad (54)$$

Using the variance functions given earlier, this takes the form

$$\left\{\bar{E}^2I^2\int_0^{l} N_i''(x)N_j''(x)\,\mathrm{d}x\cdot\int_0^{l} N_r''(x)N_s''(x)\,\mathrm{d}x\right\}$$
$$\cdot\frac{\sigma_{\mathrm{E}}^2}{2}\left[L_0^2\gamma_{\mathrm{E}}(L_0)-L_1^2\gamma_{\mathrm{E}}(l_1)+L_2^2\gamma_{\mathrm{E}}(L_2)-L_3^2\gamma_{\mathrm{E}}(L_3)\right], \quad (55)$$

where the lengths L_0, L_1, L_2, L_3 are indicated in Fig. 2.

In terms of correlation functions, the above equation is written for $L_i = U_i$, in the following manner:

$$\langle k_{ij}(\Omega)\cdot k_{rs}(\Omega)\rangle = \left\{\bar{E}^2I^2\int_0^{l_i} N_i''(x)N_j''(x)\,\mathrm{d}x\cdot\int_0^{l_i} N_r''(x)N_s''(x)\,\mathrm{d}x\right\}$$
$$\cdot\frac{\sigma_E^2}{2}\left[U_0^2\int_0^{U_0}\int_0^{U_0}\rho_a(\xi_1-\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2-U_1^2\int_0^{U_1}\int_0^{U_1}\rho_a(\xi_1-\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\right.$$
$$\left.+U_2^2\int_0^{U_2}\int_0^{U_2}\rho_a(\xi_1-\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2-U_3^2\int_0^{U_3}\int_0^{U_3}\rho_a(\xi_1-\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\right]. \quad (56)$$

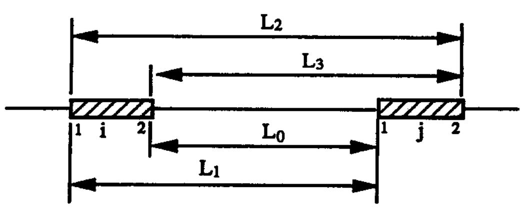

Fig. 2. Correlation between the coefficients corresponding to two arbitrarily located finite elements i and j.

Similar expressions can be written for $\langle m_{ij}(\Omega)\cdot m_{rs}(\Omega)\rangle$, for stiffness and mass elements which correspond to a node shared by two elements, as one can see from the above, the variances and covariances automatically take care of the superposition law.

7. EIGENSOLUTION STATISTICS

The eigenvalue problem is written as

$$[K^{\text{tot}}]\{x\} = \lambda[M]\{x\}, \tag{57}$$

$$[\bar{K}^{\text{tot}} + K^{\text{tot}}(\Omega)]\{x\} = \lambda[\bar{M} + M(\Omega)]\{x\}, \tag{58}$$

where

$$[K^{\text{tot}}] = -[K] + P[K_{GC} + \alpha_0[K_{CNC1}]] \text{ for Beck's column} \tag{59}$$

and

$$[K^{\text{tot}}] = -[K] + Q[K^*_{GC} + \alpha_D[K_{DNC}]] \text{ for Leipholz column.} \tag{60}$$

Moreover,

$$[\bar{K}^{\text{tot}}] = -[\bar{K}] + P[K_{GC} + \alpha_0[K_{CNC1}]] \tag{61}$$

and

$$K^{\text{tot}}(\Omega) = -[K(\Omega)], \text{ for Beck's column} \tag{62}$$

and

$$[\bar{K}^{\text{tot}}] = -[\bar{K}] + Q[K^*_{GC} + \alpha_D[K_{DNC}]] \tag{63}$$

and

$$K^{\text{tot}}(\Omega) = -[K(\Omega)] \text{ for Leipholz column.} \tag{64}$$

$\bar{K}^{\text{tot}}$, $\bar{M}$ are the deterministic matrices and $K^{\text{tot}}(\Omega)$, $M(\Omega)$ are the random matrices which arise from the fluctuation processes $a(x)$, $b(x)$, respectively. Since the material property distributions are interpreted to be one-dimensional univariate stochastic fields, the random components of stiffness and mass matrices result from local (spatial) averaged quantities. Now, consider a random eigenvalue problem described by

$$[K^{\text{tot}}]\{x\} = \lambda[M]\{x\}, \tag{65}$$

where elements of K^{tot}, M are random variables.

The perturbations of the eigenvalues λ_i can be shown to be

$$\mathrm{d}\lambda_i = \sum_{r=1}^{n}\sum_{s=1}^{n} \frac{\partial \lambda_i}{\partial k^{\text{tot}}_{rs}}\,\mathrm{d}k^{\text{tot}}_{rs} + \sum_{r=1}^{n}\sum_{s=1}^{n} \frac{\partial \lambda_i}{\partial m_{rs}}\,\mathrm{d}m_{rs}, \tag{66}$$

where n is the order of matrices $[K^{\text{tot}}]$ and $[M]$.

Since the K^{tot} and M elements are regular functions of the influence coefficients, if one seeks the derivatives of each eigenvalue about the averaged eigenvalue, one can get,

$$\frac{\partial \lambda_i}{\partial k_{rs}^{\text{tot}}} = y_{ri} x_{si} / y_i^{\mathrm{T}} M x_i, \tag{67}$$

$$\frac{\partial \lambda_i}{\partial m_{rs}} = -\lambda_i (y_{ri} x_{si} / y_i^{\mathrm{T}} M x_i). \tag{68}$$

In the above, y_i is the left eigenvector and x_i is the right eigenvector, so that

$$([K^{\text{tot}}] - \lambda_i [M])\{x\} = 0 \tag{69}$$

and

$$\{y_i\}^{\mathrm{T}}([K^{\text{tot}}] - \lambda_i [M]) = 0. \tag{70}$$

If the elements of random matrices result from continuous stochastic processes describing material property fluctuations, for sufficiently small perturbations, one can write,

$$\begin{aligned}
\lambda_r &= \bar{\lambda}_r + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \lambda_r}{\partial k_{ij}^{\text{tot}}} (k_{ij}^{\text{tot}} - \bar{k}_{ij}^{\text{tot}}) + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \lambda_r}{\partial m_{ij}} (m_{ij} - \bar{m}_{ij}) \\
&= \bar{\lambda}_r + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \lambda_r}{\partial k_{ij}^{\text{tot}}} (k_{ij}^{\text{tot}}(\Omega)) + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \lambda_r}{\partial m_{ij}} (m_{ij}(\Omega)).
\end{aligned} \tag{71}$$

Therefore, the mean values are

$$\langle \lambda_r \rangle = \langle \bar{\lambda}_r \rangle + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \lambda_r}{\partial k_{ij}^{\text{tot}}} \langle k_{ij}^{\text{tot}}(\Omega) \rangle + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \lambda_r}{\partial m_{ij}} \langle m_{ij}(\Omega) \rangle. \tag{72}$$

As $\langle k_{ij}^{\text{tot}}(\Omega) \rangle = \langle m_{ij}(\Omega) \rangle = 0$

$$\langle \lambda_r \rangle = \bar{\lambda}_r. \tag{73}$$

The covariance between any two eigenvalues is given by

$$\begin{aligned}
\langle (\lambda_p - \bar{\lambda}_p)(\lambda_q - \bar{\lambda}_q) \rangle &= \left\langle \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \lambda_p}{\partial k_{ij}^{\text{tot}}} (k_{ij}^{\text{tot}}(\Omega)) + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \lambda_p}{\partial m_{ij}} (m_{ij}(\Omega)) \right\} \right. \\
&\quad \left. \cdot \left\{ \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{\partial \lambda_q}{\partial k_{rs}^{\text{tot}}} (k_{rs}^{\text{tot}}(\Omega)) + \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{\partial \lambda_q}{\partial m_{rs}} (m_{rs}(\Omega)) \right\} \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{\partial \lambda_p}{\partial k_{ij}^{\text{tot}}} \cdot \frac{\partial \lambda_q}{\partial k_{rs}^{\text{tot}}} \langle k_{ij}^{\text{tot}}(\Omega) k_{rs}^{\text{tot}}(\Omega) \rangle \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{\partial \lambda_p}{\partial m_{ij}} \cdot \frac{\partial \lambda_q}{\partial m_{rs}} \langle m_{ij}(\Omega) m_{rs}(\Omega) \rangle,
\end{aligned} \tag{74}$$

since convariance $(k_{ij}^{\text{tot}}(\Omega) m_{rs}(\Omega)) = 0$.

The variance is therefore given by

$$\begin{aligned}
\sigma_p^2 = \operatorname{var}(\lambda_p) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{\partial \lambda_p}{\partial k_{ij}^{\text{tot}}} \frac{\partial \lambda_p}{\partial k_{rs}^{\text{tot}}} \langle k_{ij}^{\text{tot}}(\Omega) k_{rs}^{\text{tot}}(\Omega) \rangle \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{\partial \lambda_p}{\partial m_{ij}} \frac{\partial \lambda_p}{\partial m_{rs}} \langle m_{ij}(\Omega) m_{rs}(\Omega) \rangle.
\end{aligned} \tag{75}$$

Since the stiffness and mass matrices are derived using stochastic finite element method consisting of a deterministic component and zero mean fluctuating components, the mean value of any random stiffness or mass matrix element is given by the corresponding components of the deterministic parts of those random matrices. Therefore,

$$\bar{k}_{ij}^{\text{tot}} = -\bar{E}I \int_0^{l_c} N_i''(x) \cdot N_j''(x)\,\mathrm{d}x + P[k_{ij}^{GC} + \alpha_0 k_{ij}^{\text{CNC1}}], \quad \text{for Beck's column} \tag{76}$$

and

$$\bar{k}_{ij}^{\text{tot}} = -\bar{E}I \int_0^{l_c} N_i''(x) \cdot N_j''(x)\,\mathrm{d}x + Q[k_{ij}^{GC*} + \alpha_D k_{ij}^{\text{DNC}}], \quad \text{for Leipholz column} \tag{77}$$

and

$$\bar{m}_{ij} = \bar{m} \int_0^{l_c} A N_i(x) N_j(x)\,\mathrm{d}x. \tag{78}$$

Since $\operatorname{Var}(k_{ij}^{\text{tot}}) = \operatorname{Var}(k_{ij}(\Omega))$, the covariance between two stiffness elements is given by eqn (53) and an identical expression can be written between two mass matrix elements. Hence, the covariance matrix between stiffness and mass elements can be found. That is,

$$[c_{km}] = \begin{bmatrix} \operatorname{Var}(k_1) & \operatorname{COV}(k_1,k_2) & \cdots & \operatorname{COV}(m_1,k_1) & \cdots & \operatorname{COV}(m_{n2},k_1) \\ \operatorname{COV}(k_1,k_2) & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \operatorname{COV}(m_1,k_1) & \operatorname{COV}(m_1,k_2) & \cdots & \operatorname{VAR}(m_1) & \cdots & \operatorname{COV}(m_{n2},m_1) \\ \operatorname{COV}(m_{n2},k_1) & \operatorname{COV}(m_{n2},k_2) & \cdots & \operatorname{COV}(m_{n2},m_1) & \cdots & \operatorname{VAR}(m_{n2}) \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad \text{where } n2 \text{ represents } n^2. \tag{79}$$

In the above matrix, the submatrices C_{12}, C_{21} become null matrices as the two material property variations are independent stochastic fields. From this, the covariance between two k_{ij}^{tot} elements is calculated.

As a result, the mean values of eigenvalues become the eigenvalues obtained by solving the unperturbed eigenvalue problem,

$$[\bar{K}^{\text{tot}}]\{\bar{x}_i\} = \bar{\lambda}_i[\bar{M}]\{\bar{x}_i\}, \tag{80}$$

where $[\bar{K}^{\text{tot}}]$, $[\bar{M}]$ are formed using the deterministic components of k_{ij}^{tot} and m_{ij}.

Then the variances of the eigenvalues are given by

$$\sigma_p^2 = \operatorname{var}(\lambda_p) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n} \frac{\partial\lambda_p}{\partial k_{ij}^{\text{tot}}}\frac{\partial\lambda_p}{\partial k_{rs}^{\text{tot}}} \operatorname{COV}(k_{ij}^{\text{tot}}(\Omega)k_{rs}^{\text{tot}}(\Omega))$$
$$+ \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n} \frac{\partial\lambda_p}{\partial m_{ij}}\frac{\partial\lambda_p}{\partial m_{rs}} \operatorname{COV}(m_{ij}(\Omega)m_{rs}(\Omega)) \tag{81}$$

and the covariance between any two eigenvalues is given by

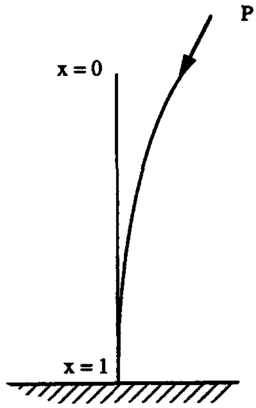

Fig. 3. Beck's column—example.

$$\mathrm{COV}(\lambda_p, \lambda_q) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n} \frac{\partial \lambda_p}{\partial k_{ij}^{\mathrm{tot}}} \frac{\partial \lambda_q}{\partial k_{rs}^{\mathrm{tot}}} \mathrm{COV}(k_{ij}^{\mathrm{tot}}(\Omega) k_{rs}^{\mathrm{tot}}(\Omega))$$
$$+ \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n} \frac{\partial \lambda_p}{\partial m_{ij}} \frac{\partial \lambda_q}{\partial m_{rs}} \mathrm{COV}(m_{ij}(\Omega) m_{rs}(\Omega)). \quad (82)$$

Similar expressions can be written to evaluate the covariance of eigenvector elements. The covariance matrix of the eigenvalues and eigenvectors can then be constructed. Depending on the boundary conditions of the structure being analysed corresponding rows and columns can be removed from the $[K^{\mathrm{tot}}]$, $[M]$ matrices and the same thing can be done with the covariance matrix between the stiffness elements and mass elements. Summations are accordingly reduced in terms of indices, i.e. n corresponds to the unrestrained global degree of freedom of the total structure being analysed.

At this point, it may be of interest to compare the above procedure with that adopted by Liaw and Yang (1991) in a similar study. As mentioned earlier the above-referenced work deals only with random variables and not random fields as considered in the present paper. Further, the perturbation series is of the form

$$\lambda = \lambda^{(0)} + \sum_{r=1}^{m} \lambda_r^{(1)} \alpha_r + \cdots$$

where α_r are the random variables and the expectation $E[\alpha_r \alpha_s]$ is assumed to be explicitly known as input information. As has been already pointed out, the use of random variables together with a perturbation series in the above form makes a reliability evaluation possible by a straightforward extension of the procedure for response moment evaluation. For the evaluation of second or higher order statistics of eigenvalues and eigenvectors using second order analysis, availability of higher order moments of input random variable is a requirement. While such information may not be available for non-Gaussian cases, its calculation for Gaussian cases is very tedious or in most cases impossible as outlined by Shinozuka and Yamazaki (1988). The superiority of the method developed in the present paper, in this respect, in as much as it employs a random field representation and sensitivity vectors leading directly to the covariance matrix of output quantities is worthy of special mention.

8. NUMERICAL EXAMPLE

The Beck's column as shown in Fig. 3 is taken as an example. The Young's modulus is considered to be stochastically distributed. The material property values are taken as:

$$\bar{E} = 2.1 \times 10^5\ \mathrm{N\,mm^{-2}} \quad \text{and} \quad m = 7.83 \times 10^{-9}\ \mathrm{N\,sec\,mm^{-4}},$$

and further the length $= 7.35$ m; cross-sectional area $= 4329\ \mathrm{mm}^2$; moment of inertia $=$

Table 1. Variances of vibration frequencies of Beck's column with uncorrelated E values for $P/P_{fl} = 0.25$

Input variance	var λ_1	var λ_2	var λ_3	var λ_4	var λ_5	var λ_6
0.01	23.5708	0.4218	0.0772	4.0229	100.3446	24.7048
0.02	47.1417	0.8436	0.1545	8.0457	200.6892	49.4097
0.03	70.7125	1.2654	0.2317	12.0686	301.0337	74.1145
0.04	94.2833	1.6872	0.3089	16.0914	401.3783	98.8195
0.05	117.8541	2.1090	0.3862	20.1143	501.7229	123.5242
0.06	141.4250	2.5308	0.4634	24.1372	602.0675	148.2291
0.07	164.9958	2.9526	0.5406	28.1600	702.4210	172.9340
0.08	188.5666	3.3744	0.6179	32.1829	802.7566	197.3888
0.09	212.1375	3.7962	0.6951	36.2057	903.1012	222.3436
0.10	235.7083	4.2180	0.7723	40.2286	1003.4458	247.0485
Mean value	(11340)	(48850)	(215500)	(1107000)	(3944000)	(15760000)

Table 2. Covariance matrix of eigenvalues for $P/P_{fl} = 0.25$ and input variance = 0.01

$$\begin{bmatrix} 23.5708 & -48.1849 & 9.6245 & 1.2205 & -2.8253 & -12.3817 \\ & 0.4218 & -19.1673 & -2.1016 & 5.7389 & 24.4282 \\ & & 0.0772 & 0.4812 & -1.2195 & -2.9141 \\ \text{Symmetric} & & & 4.0029 & -0.1422 & -0.4426 \\ & & & & 100.3446 & 0.8862 \\ & & & & & 24.7048 \end{bmatrix}$$

2.672×10^7 mm^4 and the tip axial load $P = 0.25P_{fl}$ (the tip axial load is taken to be 0.25 times the flutter load P_{fl} of the Beck's column in this example).

The stochastic process $a(x)$ representing the fluctuating components of elastic modulus is represented by an exponential type correlation function relationship. This is the first-order autoregressive or Markov process representation. Here, the correlation function is given by

$$\rho(\xi) = e^{-|\xi|/c}, \text{ where } c = \text{constant, and } \xi = \text{separation distance.}$$

Further, the variance function is given by

$$\gamma(U) = 2\left(\frac{c}{U}\right)^2\left(\frac{U}{c} - 1 + e^{-U/c}\right).$$

For $U \to \infty$, or for large finite element sizes, the above function can be approximated by $\rho(U) = 2c/U$.

So, the covariance between stiffness elements and mass elements is zero and that between stiffness elements and geometric stiffness elements, as one knows, is also zero.

The NAG routines are used for solving the eigensystems. The variance of the fundamental eigenvalue as well as the covariance between the eigenvalues for different values of input variance are listed in Tables 1 and 2.

9. CONCLUSIONS

A new version of stochastic finite element method is developed herein to solve the more general non-self-adjoint eigenvalue problems. The method developed has been demonstrated through solving the free-vibration problem of Beck's and Leipholz columns which have uncertain material property variations. Uncertain material property values like Young's modulus and mass density are modeled using stochastic fields and not by using random variables. An efficient version of the stochastic finite element method has been developed by the present authors for self-adjoint systems (Ramu and Ganesan, 1991, 1992a,b) and singularity problems (Sankar et al., 1992b). Also its efficiency has been clearly expained in detail in those works. Further, it has already been extended to gyroscopic non-self-adjoint systems through a Hamiltonian formulation (Sankar *et al.*, 1992a). For non-self-adjoint systems arising in the case of non-conservatively loaded structures, it is well known that a unique scalar work function is not available so that such a Hamiltonian formulation can be used. So, the method developed for gyroscopic systems by the present authors (Sankar *et al.*, 1992a) has been modified herein by integrating it with a virtual work formulation and sensitivity gradients for non-self-adjoint systems. The external term in the conventional form of Hamilton's principle is interpreted to be the virtual work done by the non-conservative forces. Combining this strategy with an asymptotic expansion employing eigensolution gradients based on adjoint system concepts, the free vibration problem of non-conservatively loaded stochastic columns is solved for response moments. Further, the features of the method developed for self-adjoint systems have been preserved. As a result, the advantages and superiority of the method developed by the present authors for self-adjoint systems become applicable to the present formulation also. Among them, the major ones like the rigorous representation of material property variation by stochastic fields and the capability of making the discretization process independent of system uncertainties, are worthy of special mention here. Thus, the response moment calculations are made in a more straightforward manner without converting the stochastic fields into a set of random variables. At this point, it may be noted that estimation of reliability of the system would call for a totally different type of analysis (Benaroya and Rehak, 1988) though more complex, since the uncertainties both of the system parameters and the loads have to be accounted for.

In the numerical example, the first-order autoregressive model (most commonly known as the Markov model) has been employed which is the most commonly observed correlation model for the field data of material property variation, loading uncertainty, etc. The present work addresses the free vibration problem of non-conservatively loaded stochastic columns. According to the dynamic stability theory, the free vibration problem is to be solved first to obtain the critical loads of non-conservative columns. The frequency equation relates the free vibration frequencies with the critical loads. Once the statistics of natural frequencies are known, corresponding to the critical loads of the averaged column, they can be transformed to yield the statistics of critical loads through the nonlinear frequency equation. Such a procedure has already been developed and illustrated using extensive numerical examples by the present authors (Ramu and Ganesan, 1992c) and so is not repeated here. However, statistics of critical loads can also be obtained using the finite element matrices that are developed in the present work.

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